

ON THE LOCAL STRUCTURE OF QUANTIZATIONS IN CHARACTERISTIC p

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ABSTRACT. Let A be a central quantization of an affine Poisson variety X over a field of characteristic $p > 0$. We show that the completion of A with respect to a closed point $y \in X$ is isomorphic to the tensor product of the Weyl algebra with a local Poisson algebra. This result can be thought of as a positive characteristic analogue of results of Losev and Kaledin about slice algebras of quantizations in characteristic 0.

Let \mathbf{k} be an algebraically closed field of characteristic $p > 0$. By a Poisson algebra over \mathbf{k} we will understand a commutative \mathbf{k} -algebra equipped with a \mathbf{k} -linear Poisson bracket $\{ \ , - \}$. Throughout given a Poisson algebra B and $b \in B$, by $\text{ad}(b)$ we will denote the corresponding derivation $\{b, \ \}$. While for a non-commutative algebra A and its element $a \in A$, by $\text{ad}(a)$ we will denote the commutator $[a, \ \]$. Also, for an algebra A by $Z(A)$ we will denote its center. By a (deformation) quantization of a Poisson algebra $(B, \{ \ , - \})$ we will understand an associative $\mathbf{k}[[\hbar]]$ -algebra A such that A is topologically free $\mathbf{k}[[\hbar]]$ -module, $B = A/\hbar A$, and

$$\{a \bmod \hbar, b \bmod \hbar\} = \left(\frac{1}{\hbar}[a, b]\right) \bmod \hbar, \quad a, b \in A.$$

Following [BK], we will give the following

Definition 0.1. A quantization A of a Poisson \mathbf{k} -algebra B is called weakly central if for any $a \in A$ there exists $a^{[p]} \in A$ such that

$$a^p - \hbar^{p-1}a^{[p]} \in Z(A).$$

In particular $Z(A) \bmod \hbar$ contains B^p .

Existence of a weakly central quantization of a Poisson algebra B imposes on it the following necessary condition: for any $b \in B$ there exists $b^{[p]} \in B$ such that $\text{ad}(b)^p = \text{ad}(b^{[p]})$. This motivates the following

Definition 0.2. A Poisson \mathbf{k} -algebra B is said to be a weakly restricted Poisson algebra if for any $f \in B$ there exists $f^{[p]} \in B$ such that

$$\text{ad}(f)^p = \text{ad}(f^{[p]}).$$

Our goal is to study the local structure of a quantization algebra A in terms of symplectic leaves of the affine Poisson variety $\text{Spec } B$. Let us recall the definition of symplectic leaves in algebraic setting. A symplectic leaf in an affine Poisson variety $X = \text{Spec } B$ is a locally closed subvariety $Y \subset X$

such that the corresponding ideal of its closure $I(\bar{Y})$ is a Poisson ideal and Y is a smooth symplectic variety under the Poisson bracket induces from B .

By $\overline{W_{n,\hbar}}$ we will denote the complete Weyl algebra over $\mathbf{k}[[\hbar]]$:

$$\overline{W_{n,\hbar}} = \mathbf{k}[[\hbar]]\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle, \quad [x_i, x_j] = 0 = [y_i, y_j], [x_i, y_j] = \hbar \delta_{ij}$$

Similarly W_n will denote the usual Weyl algebra

$$W_n = \mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_n], \quad [x_i, x_j] = 0 = [y_i, y_j], [x_i, y_j] = \delta_{ij}$$

Clearly W_n is a subalgebra of $\overline{W_{n,\hbar}}[\hbar^{-1}]$.

We will recall couple of simple lemmas that will be used in the proof of our main result.

Lemma 0.3. *Let S be a quantization of a Poisson algebra over \mathbf{k} , let $H \subset S$ be a $\mathbf{k}[[\hbar]]$ -subalgebra which is isomorphic to a quotient of $\overline{W_{n,\hbar}}$. Assume that $Z(H) \subset Z(S)$. Let S' be the centralizer of H in S . Then the multiplication map $H \otimes_{Z(H)} S' \rightarrow S$ is an isomorphism. Moreover if $I \subset S$ is a two-sided ideal closed under $\frac{1}{\hbar} \text{ad}(a), a \in S$, then $I = H \otimes_{Z(H)} I'$, where $I' = S' \cap I$.*

Proof. Proceeding by induction on n , it suffices to show the statement for $n = 1$. Let H be generated over $\mathbf{k}[[\hbar]]$ by elements x, y where $[x, y] = \hbar$. By the assumption $x^p, y^p \in Z(S)$.

Let us view S as a module over the Weyl algebra

$$W_1 = \mathbf{k}\langle \alpha, D \rangle, [\alpha, D] = 1,$$

where the action by α is just the left multiplication by x , and $D(a) = \frac{1}{\hbar}[y, a], a \in S$. Then $D^p(S) = 0$. Therefore S is a left $W_1(\mathbf{k})/(D^p)$ -module. On the other hand since $W_1/(D^p)$ is isomorphic to the $p \times p$ matrix algebra over $\mathbf{k}[\alpha^p]$, it follows that $S = k[x] \otimes_{\mathbf{k}[\alpha^p]} S'$, where S' is the centralizer of y in S . Then S' is $\mathbf{k}[y]$ -module closed under the bracket $\frac{1}{\hbar}[x, -]$. Arguing similarly as above we get that $S = H \otimes_{Z(H)} S_1$, where S_1 is the centralizer of H . Similarly $I = H \otimes_{Z(H)} I'$, where $I' = I \cap S'$. □

Lemma 0.4. *Let A be a weakly central quantization. Let $f, g \in A$ be such that $[f, g] = \hbar$. Then*

$$(f + g^{p-1}f^{[p]})^p - (g^{p-1}f^{[p]})^p \in Z(A).$$

Proof. Recall that in the Weyl algebra over a commutative ring R , $W_1(R) = \mathbf{k}[R]\langle x, y \rangle / ([x, y] - 1)$ the following relation holds (see for example [[B], Theorem 1.3])

$$(x + ry^{p-1})^p = x^p + (ry^{p-1})^p - r, r \in R.$$

In our case since $[f, \frac{1}{\hbar}g] = 1$ and $f^{[p]}$ commutes with f, g and since $f^p - \hbar^{p-1}f^{[p]} \in Z(A)$, using the above equality we obtain the desired result. □

We have the following key result.

Proposition 0.5. *Let (B, m) be a complete Noetherian Poisson algebra with the maximal ideal m such that B is a finitely generated module over B^p . Let $I \subset m$ be a Poisson ideal. Let A be a weakly central quantization of B . Assume that $S = B/I$ contains elements $x_i, y_j, 1 \leq i, j \leq n$ such that*

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0, \quad \text{ad}(x_i)^p = \text{ad}(y_j)^p = 0.$$

Then there are elements $z_i, w_j \in A, 1 \leq i, j \leq n$ such that they lift x_i, y_j and

$$[z_i, w_j] = \delta_{ij}\hbar, \quad [z_i, z_j] = [w_i, w_j] = 0, \quad z_i^p, w_j^p \in Z(A).$$

Here is the Poisson version of this proposition without quantizations. Its proof is essentially the same as one of Proposition 0.5.

Proposition 0.6. *Let (B, m) be a complete weakly restricted Noetherian Poisson algebra with the maximal ideal m such that B is a finitely generated module over B^p . Let $I \subset m$ be a Poisson ideal. Assume that $S = B/I$ contains elements $x_i, y_j, 1 \leq i, j \leq n$ such that*

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0, \quad \text{ad}(x_i)^p = \text{ad}(y_j)^p = 0.$$

Then there are elements $z_i, w_j \in B, 1 \leq i, j \leq n$ such that they lift x_i, y_j and

$$\{z_i, w_j\} = \delta_{ij}, \quad \{z_i, z_j\} = \{w_i, w_j\} = 0, \quad \text{ad}(z_i)^p = \text{ad}(w_j)^p = 0.$$

Proof of Proposition 0.5. We will proceed by induction on n . Let $p : A \rightarrow A/\hbar A = B$ denote the quotient map. Put $m' = p^{-1}(m)$. Thus m' is the maximal ideal of A . Put $m'^{[p]} = \{x^p, x \in m'\}$. Let us put $J = p^{-1}(I) + m'^{[p]}A$. Then J is a two sided ideal in $A, \hbar \in J$. Moreover for any $a \in A$ we have $\text{ad}(a)(J) \subset J$. Let $f_1, g_1 \in A$ be arbitrary lifts of x_1, y_1 respectively. Therefore

$$[f_1, g_1] = \hbar + \hbar z_1, \quad z_1 \in J.$$

Now given elements $f_n, g_n \in A, n \geq 1$ such that

$$x_1 = f_n \mod p^{-1}(I), y_1 = g_n + p^{-1}(I), \quad [f_n, g_n] = \hbar + \hbar z_n, z_n \in J^n,$$

we will construct $f_{n+1}, g_{n+1} \in A$ such that

$$\begin{aligned} x_1 &= f_{n+1} \mod p^{-1}(I), \quad y_1 = g_{n+1} \mod p^{-1}(I) \\ [f_{n+1}, g_{n+1}] &= \hbar + \hbar z_{n+1}, \quad z_{n+1} \in J^{n+1}. \end{aligned}$$

Since

$$\frac{1}{\hbar}[f_n, J^m] \subset J^m, \quad \frac{1}{\hbar}[g_n, J^m] \subset J^m, m \geq 1.$$

We may write

$$\hbar z_n = \hbar f_n^{p-1} z'_n + [z''_n, g_n] + \hbar \omega,$$

for some $\omega \in J^{n+1}, z'_n, z''_n \in J^n$. Then replacing f_n by $f_n - z''_n$, we get that

$$[f_n, g_n] = \hbar + \hbar f_n^{p-1} z'_n + \hbar \omega, \quad \omega \in J^{n+1}, \quad z'_n \in J^n.$$

Now recall that by our assumption on A , for any $a \in A$ there exists $a^{[p]} \in A$ such that

$$\text{ad}(a^p) = \text{ad}(a)^p = \hbar^{p-1} \text{ad}(a^{[p]}).$$

Thus we have

$$[f_n, \hbar^{p-1} g_n^{[p]}] = \text{ad}(g_n)^{p-1}(\hbar f_n^{p-1} z'_n + \hbar \omega) = (-1) \hbar^p z'_n + \hbar^p f_n \omega''_n + \hbar^p \eta,$$

for some $\omega''_n \in J^n, \eta \in J^{n+1}$. Then

$$[f_n, g_n + \hbar f_n^{p-1} g_n^{[p]}] = \hbar + \hbar f_n^p \omega''_n + \hbar \eta', \quad f_n^p \omega'_n \in J^{n+1}, \quad \eta' \in J^{n+1}.$$

Thus we may put

$$f_{n+1} = f_n, \quad g_{n+1} = g_n + \hbar f_n^{p-1} g_n^{[p]}.$$

We see from the construction that the sequence $f_n, n \geq 1$ converges, put $f = \lim f_n$. Thus we have that

$$[f, g_n] = \hbar \mod \hbar J^n.$$

Writing $g_n = g_1 + \hbar g'_n$ and $[f, g_1] = \hbar + \hbar z_1$, we get that $z_1 = [f, -g'_n] \mod J^n$. Hence $z_1 \in \bigcap_n [f, A] + J^n$. It follows from our assumptions that A is a finitely generated module over a complete Noetherian ring $Z(A)$. Hence $[f, A]$ is finitely generated as a $Z(A)$ -module. Then it follows from the Artin-Rees lemma that there exists $g' \in A$ such that $[f, g'] = z_1$. Putting $g = g_1 - \hbar g'$ we get that

$$[f, g] = \hbar, \quad x_1 = f \mod p^{-1}(I), \quad y_1 = g \mod p^{-1}(I).$$

Next we want to modify elements f, g such that

$$\text{ad}(f)^p = \text{ad}(g)^p = 0, \quad [f, g] = \hbar.$$

We will do so inductively: let $\phi_n \in A$ be such that

$$\phi_n \mod p^{-1}(I) = x_1, \quad [\phi_n, g] = \hbar, \quad f_n^{[p]} \in m'^n.$$

We have that by Lemma 0.4

$$(\phi_n + g^{p-1} \phi_n^{[p]})^{[p]} = (g^{p-1} \phi_n^{[p]})^{[p]}.$$

But $(g^{p-1} \phi_n^{[p]})^{[p]} \in m'^{n+1}$. Indeed, this follows from the fact that if $x \in m^n$, then $x^{[p]} \in m^n$. Put $\phi_{n+1} = \phi_n + g^{p-1} \phi_n^{[p]}$. Then since $\phi_n^{[p]} \in p^{-1}(I)$, we get that $\phi_n = \phi_{n+1} \mod m'^n p^{-1}(I)$ and $\phi_{n+1}^{[p]} \in m'^{n+1}$. Putting $f = \lim \phi_n$ we get that

$$[\phi, g] = \hbar, \quad \phi^{[p]} = 0.$$

Proceeding in the same manner with g we get elements $z_1, w_1 \in A$ such that

$$[z_1, w_1] = \hbar, x_1^{[p]} = 0 = y_1^{[p]}, z_1 \mod p^{-1}(I) = x_1, w_1 \mod p^{-1}(I) = y_1.$$

Let us put

$$H = \mathbf{k}[[\hbar]] \langle z_1, w_1 \rangle, \quad \bar{z}_1 = z_1 \mod \hbar, \quad \bar{w}_1 = w_1 \mod \hbar.$$

Thus by Lemma 0.3 we have that $A = H \otimes_{Z(H)} A$, where A_1 is the centralizer of H in A . It follows that A_1 is a weakly central quantization of the Poisson centralizer of \bar{z}_1, \bar{w}_1 in B , which we denote by B_1 . Let $I_1 = p^{-1}(I) \cap A_1$. Then by Lemma 0.3 we have that $p^{-1}(I) = H \otimes_{Z(H)} I_1$. In particular,

$I_1 \bmod \hbar = I'$ is a Poisson ideal in B_1 . It follows that B_1/I' contains $x_i, y_j, i, j \geq 2$. Proceeding by induction we are done. \square

Next we will recall the following characteristic p analogue of the Darboux's theorem from [BK].

Lemma 0.7. *Let Y be an $2n$ -dimensional affine symplectic variety over \mathbf{k} such that $\mathcal{O}(Y)$ is a weakly restricted Poisson algebra. Let $y \in Y$ be a closed pint, and let $\mathcal{O}(Y)_{\bar{y}}$ denote the completion of the coordinate ring of Y with respect to y . Then $\mathcal{O}(Y)_{\bar{y}}$ is isomorphic to the Poisson algebra $\mathbf{k}[[x_1, \dots, x_n, y_1, \dots, y_n]]$ with the bracket*

$$\{x_i, x_j\} = 0 = \{y_i, y_j\}, \quad \{x_i, y_j\} = \delta_{ij}.$$

Proof. Let m denote the maximal ideal of $\mathcal{O}(Y)_{\bar{y}}$. Denote by $m^{[p]}$ the ideal generated by $b^p, b \in m$. Then it follows from results of Bezrukavnikov-Kaledin [[BK], Theorem 1.1, Proposition 3.4] that $\mathcal{O}(Y)_{\bar{y}}/m^{[p]}$ as a Poisson algebra is isomorphic to $\mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_n]/(x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p)$ with the Poisson bracket

$$\{x_i, x_j\} = 0 = \{y_i, y_j\}, \quad \{x_i, y_j\} = \delta_{ij}.$$

Now Proposition 0.6 yields the desired result. \square

Now we are ready to state and prove our main result. In what follows by a local Poisson \mathbf{k} -algebra we will mean a local \mathbf{k} -algebra equipped with a Poisson bracket such that the maximal ideal is a Poisson ideal. Let A be a weakly central quantization of an affine Poisson \mathbf{k} -variety $X = \text{Spec } B$. Let $y \in \text{Spec } B$ be a closed point, and m_y be the corresponding maximal ideal. Denote by $A_{\bar{y}}$ the completion of A with respect to two-sided ideal $p^{-1}(m_y)$ -the pre-image of m_y under the quotient $p : A \rightarrow A/\hbar A = B$. Put $p^{-1}(m_y) \cap Z(A) = Z_y$ -an ideal in $Z(A)$ containing \hbar . Denote by $Z(A)_{\bar{y}}$ the completion of $Z(A)$ with respect to Z_y . Then $A_{\bar{y}} = A \otimes_{Z(A)} Z(A)_{\bar{y}}$. Also, $A_{\bar{y}}$ is a quantization of $B_{\bar{y}}$ -the completion of B with respect to y . Let Y be a symplectic leaf in $\text{Spec } B$ containing y . Let $\dim Y = 2n$. In this setting we have the following

Theorem 0.8. *Algebra $A_{\bar{y}}$ is isomorphic to $\overline{W_{n,\hbar}} \otimes_{Z(\overline{W_{n,\hbar}})} A_{\bar{y}}^+$, where $A_{\bar{y}}^+$ is a central quantization of a complete local Poisson algebra.*

Proof. Let I be the Poisson ideal in $B_{\bar{y}}$ such that $B_{\bar{y}}/I = \mathcal{O}(Y)_{\bar{y}}$ - the completion of $\mathcal{O}(Y)$ at y . It follows that $\mathcal{O}(Y)_{\bar{y}}$ is a weakly restricted Poisson algebra. Therefore by Lemma 0.8, it follows that $\mathcal{O}(Y)_{\bar{y}} = \mathbf{k}[[x_i, y_j]], 1 \leq i, j \leq n$ with the usual Poisson bracket

$$\{x_i, x_j\} = 0 = \{y_i, y_j\}, \quad \{x_i, y_j\} = \delta_{ij}.$$

Now Proposition 0.5 and Lemma 0.7 imply that $A_{\bar{y}} = \overline{W_{n,\hbar}} \otimes_{Z(\overline{W_{n,\hbar}})} A_{\bar{y}}^+$, where $A_{\bar{y}}^+$ is the centralizer of $W_{n,\hbar}$ in $A_{\bar{y}}, \hbar \in A_{\bar{y}}^+$. Put $L = A_{\bar{y}}^+/\hbar A_{\bar{y}}^+ \subset B_{\bar{y}}$.

Then $A_{\bar{y}}^+$ is a quantization of L and $B_{\bar{y}} = \widehat{\mathcal{O}(Y)_{\bar{y}}} \otimes_{\widehat{\mathcal{O}(Y)_{\bar{y}}}} L$, where $\widehat{\mathcal{O}(Y)_{\bar{y}}} \subset B_{\bar{y}}$, $\widehat{\mathcal{O}(Y)_{\bar{y}}} = \overline{W_{n,\hbar}}/\hbar$ is a lift of $\mathcal{O}(Y)_{\bar{y}}$ under the quotient map $B_{\bar{y}} \rightarrow \mathcal{O}(Y)_{\bar{y}}$. Thus L is the Poisson centralizer of $\widehat{\mathcal{O}(Y)_{\bar{y}}}$ in $B_{\bar{y}}$. Put $m' = \overline{m_y} \cap L$. Then m' is the maximal ideal of L . We claim that m' is in fact a Poisson ideal. Indeed the image of m' in $B_{\bar{y}}/I = \mathcal{O}(Y)_{\bar{y}}$ is in the center, therefore

$$\{m', m'\} \subset I \cap L \subset m'.$$

Thus $A_{\bar{y}}^+$ is a quantization of a complete local Poisson algebra L as desired. \square

Remark 0.9. This result is motivated and may be seen as a positive characteristic analogue of results of Kaledin [[K], Proposition 3.3] and Losev [L] on slice algebras of quantizations over \mathbb{C} .

As an immediate corollary of Theorem 0.8, we can reprove the following Kac-Weisfeiler type statement for quantizations as in [T].

Corollary 0.10. *Let M be an A -module which is finite and free over $\mathbf{k}[[\hbar]]$, such that $M/\hbar M$ is supported on y . Then $\dim_{\mathbf{k}[[\hbar]]} M$ is a multiple of p^n .*

Proof. It follows that $M[\hbar^{-1}]$ is a module over $A_{\bar{y}}[\hbar^{-1}]$. Therefore, by Theorem 0.8 $M[\hbar^{-1}]$ is a module over $\overline{W_n}[\hbar^{-1}]$. Hence $\dim_{\mathbf{k}[[\hbar]]} M$ is divisible by p^n . \square

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